

Fractal-Based Point Processes

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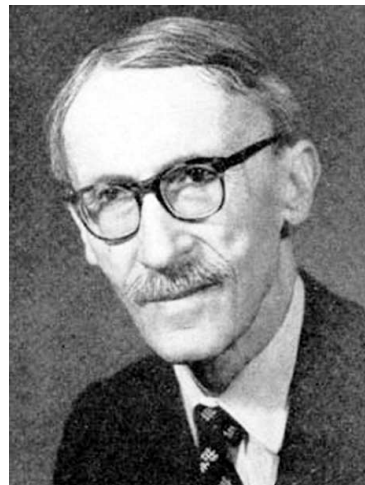
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Processes Based on the Alternating Fractal Renewal Process



Carl Friedrich Gauss (1777–1855), a celebrated German mathematician, arrived at the limit theorem for random variables that bears his name.



The French mathematician **Paul Lévy (1886–1971)** developed a family of “stable” probability distributions and processes that have widespread applicability.

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As implied by its name, the **alternating renewal process** $X(t)$ alternates between two values, a and b , as portrayed in Fig. 8.1. We set these values to unity and zero, respectively, in accordance with the usual convention, dictated by algebraic convenience (we exclude the degenerate case $a = b$). The results for these particular values of a and b are linked to those for general values by straightforward linear transforms. The dwell times in the upper and lower states derive from two separate distributions; they are also taken to be independent, thereby endowing the process with its renewal character.

Because of its apparent similarity to a Morse-code sequence, many authors call the bistable step waveform associated with the alternating renewal process a **random telegraph signal** (RTS) (others use the appellation **on–off process**). Gleason Willis Kenrick provided its correlation function and spectrum in 1929.¹ Rice (1944, 1945) subsequently studied this process extensively, with a particular emphasis on transition times that follow a Poisson process (exponentially distributed dwell times).

The alternating renewal process often serves as a useful mathematical model for describing **burst noise** in semiconductor devices (Machlup, 1954). Noise with these characteristics, first observed in junction transistors (Montgomery, 1952), also afflicts other kinds of electronic devices and integrated circuits under certain conditions (see, for example, Buckingham, 1983, Chapter 7). It is often ascribed to the presence of microplasmas, defects such as crystallographic dislocations, or to the on–off switching associated with various conduction paths. The random telegraph signal enjoys a broad variety of other applications.

A special case of the alternating renewal process is the **alternating fractal renewal process**, in which one or both of the dwell-time distributions follows a fractal (power-law) form, as indicated in the caption to Fig. 8.1. This process differs from the

¹ Kenrick's work was perhaps the first to use the correlation function of a random process to determine its spectrum. The Fourier-transform relation between these two quantities later came to be called the Wiener–Khinchin theorem (Wiener, 1930; Khinchin, 1934).

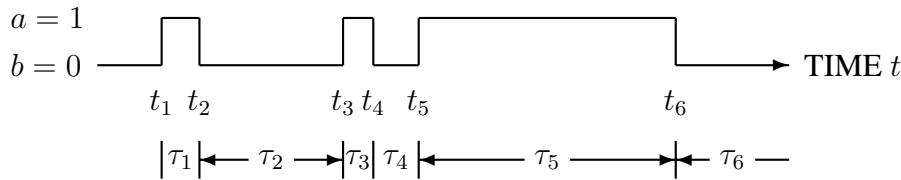


Fig. 8.1 The *alternating renewal process* switches between two values, chosen for convenience to be $a = 1$ and $b = 0$. In general, the odd and even interevent intervals derive from two different distributions. All interevent intervals are independent. In the *alternating fractal renewal process*, one or both of the dwell-time distributions follows a fractal (power-law) form.

fractal renewal process considered in Chapter 7 by the presence of two time-interval distributions and by the fact that the process is finite-valued rather than composed of Dirac δ functions. The power-law-distributed dwell times lead to a $1/f$ -type spectrum (Lowen & Teich, 1993d).

Various forms of this process serve as plausible models for semiconductor $1/f$ -type noise arising from fluctuations associated with the capture and release of carriers at traps (McWhorter, 1957; Stepanescu, 1974; Buckingham, 1983; Lowen & Teich, 1992b; Sikula, 1995). It also proves useful for characterizing the behavior of individual ion channels in biological membranes (Schick, 1974; Lowen & Teich, 1993d,c; Liebovitch et al., 2001). This process finds its place in other arenas such as dynamics in systems with fractal boundaries (Aizawa & Kohyama, 1984; Aizawa, 1984; Arecchi & Lisi, 1982; Arecchi & Califano, 1987; Sapoval, Baldassarri & Gabrielli, 2004), fluorescence fluctuations of nanoparticles diffusing through a region of focused laser light (Zumofen, Hohlbein & Hübner, 2004), the analysis of rainfall data (Schmitt et al., 1998), computer network traffic (as discussed in Chapter 13), and the generation of fractal test signals (as considered in Prob. 8.6). After considering the statistical properties of the alternating renewal process, we turn to the properties of sums thereof. The result is binomial noise, which, as the number of constituent processes increases without limit, converges to a Gaussian form. Similarly, the sum of alternating *fractal* renewal processes gives rise to fractal binomial noise, which, in turn, converges to the fractal Gaussian process discussed in Sec. 6.3.3 (Lowen & Teich, 1993d, 1995). This property provides a simple and plausible rationale for the fractal Gaussian behavior of nerve-membrane voltage fluctuations (Verveen, 1960; Verveen & Derksen, 1968): currents flowing through individual ion channels embedded in the nerve membrane behave as on-off processes with intervals between channel openings and closings that are power-law distributed. The sum of large numbers of these converges to a fractal Gaussian process.

Sums of independent and identically distributed random variables with *finite* second moments converge to normal form via Gauss' (1809) **central limit theorem**. There is a widespread perception that the central limit theorem always holds although

this is, of course, not the case.² Independent and identically distributed constituent variables with *infinite* second moments converge instead to the family of stable distributions set forth by Paul Lévy (1940, 1954; see also Samorodnitsky & Taqqu, 1994; Bertoin, 1998; Sato, 1999). This is sometimes called the **noncentral limit theorem**. The associated distributions have power-law tails and differ dramatically in character from the normal distribution.

Finally, we briefly consider the properties of doubly stochastic and integrate-and-reset point processes driven by fractal binomial noise (Lowen & Teich, 1993b, 1995; Thurner et al., 1997); these constructs find particular use in neuroscience because of the prevalence of fractal binomial noise and fractal Gaussian processes in neurobiology.

8.1 ALTERNATING RENEWAL PROCESS

The dwell times in the $X(t) = a = 1$ state, corresponding to interevent intervals τ_n with odd indices n in Fig. 8.1, all derive from the same distribution. Similarly, times for which $X(t) = b = 0$, corresponding to even indices n in this figure, also share a distribution, possibly different from that describing the $X(t) = a = 1$ dwell times. All interevent intervals are independent of each other.

We use the notations τ_a and τ_b to refer to the dwell times for which $X(t) = a = 1$ and $X(t) = b = 0$, respectively. For a well-defined process in the stationary case, we require that $p_{\tau_a}(t) = p_{\tau_b}(t) = 0$ for $t < 0$, effectively prohibiting negative dwell times; and that the expected mean dwell times $E[\tau_a]$ and $E[\tau_b]$ both assume finite values. We further require that the sequence of transitions forms an orderly point process, so that Eq. (3.1) holds for the transition times of $X(t)$.

8.1.1 Amplitude statistics

The marginal moments of $X(t)$ are simple to evaluate. The expected value $E[X(t)]$ becomes simply the ratio of the mean time spent in the $X(t) = 1$ state ($E[\tau_a]$) to the mean time spent in both states ($E[\tau_a] + E[\tau_b]$). Since $X(t)$ can only take on values of zero or unity, it belongs to the family of **Bernoulli random variables** (Feller, 1968); in particular, $X^c(t) = X(t)$ for any positive real number c .

We therefore have

$$r \equiv E[X^n(t)] = \frac{E[\tau_a]}{E[\tau_a] + E[\tau_b]} \quad (8.1)$$

² As Poincaré (1908) observed with respect to the central limit theorem: “All the world believes it firmly because the mathematicians imagine that it is a fact of observation and the observers imagine that it is a theorem of mathematics.”

for all positive moments n . Further results, akin to those in Eqs. (3.4) and (3.31) include

$$\begin{aligned}
 \text{variance} \quad \text{Var}[X] &= r(1-r) \\
 \text{skewness} \quad \frac{\text{E}[(X - \text{E}[X])^3]}{\text{Var}^{3/2}[X]} &= \frac{1-2r}{\sqrt{r(1-r)}} \\
 \text{kurtosis} \quad \frac{\text{E}[(X - \text{E}[X])^4]}{\text{Var}^2[X]} - 3 &= \frac{1}{r(1-r)} - 6.
 \end{aligned} \tag{8.2}$$

8.1.2 Autocorrelation

Closed forms for the autocorrelation $R_X(t)$ do not exist in general, but rather involve an infinite sum of convolutions of arbitrary complexity (Lowen, 1992). However, simple results do obtain in the limits of very small and very large delay times t .

For small delay times, the probability of a transition between the states $X(t) = 1$ and $X(t) = 0$ becomes vanishingly small, so that

$$\begin{aligned}
 \lim_{t \rightarrow 0} R_X(t) &= \lim_{t \rightarrow 0} \text{E}[X(s) X(s+t)] = \text{E}[X^2(s)] = \text{E}[X(s)] \\
 &= r.
 \end{aligned} \tag{8.3}$$

For large delay times, on the other hand, the two values of $X(t)$ become independent, whereupon

$$\begin{aligned}
 \lim_{t \rightarrow \infty} R_X(t) &= \lim_{t \rightarrow \infty} \text{E}[X(s) X(s+t)] \\
 &= \text{E}[X(s)] \text{E}[X(s+t)] = \text{E}^2[X(s)] \\
 &= r^2.
 \end{aligned} \tag{8.4}$$

Indeed, a similar argument holds for any well-behaved real-valued process.

8.1.3 Spectrum

As shown in Sec. A.5.1, the spectrum for an arbitrary alternating renewal process takes the form (Rice, 1983; Lowen, 1992)

$$S_X(f) = \text{E}[X] \delta(f) + \frac{2(2\pi f)^{-2}}{\text{E}[\tau_a] + \text{E}[\tau_b]} \text{Re} \left\{ \frac{[1 - \phi_{\tau_a}(2\pi f)][1 - \phi_{\tau_b}(2\pi f)]}{1 - \phi_{\tau_a}(2\pi f) \phi_{\tau_b}(2\pi f)} \right\}, \tag{8.5}$$

where $\phi_{\tau_a}(\omega)$ and $\phi_{\tau_b}(\omega)$ are the characteristic functions associated with the dwell-time distributions for τ_a and τ_b , respectively.

When the means and variances exist, this spectrum approaches an asymptotic form in the low-frequency limit given by (see Sec. A.5.2),

$$\lim_{f \rightarrow 0} S_X(f) = \frac{\text{E}^2[\tau_a] \text{Var}[\tau_b] + \text{E}^2[\tau_b] \text{Var}[\tau_a]}{(\text{E}[\tau_a] + \text{E}[\tau_b])^3}, \tag{8.6}$$

whereas in the high-frequency limit we have

$$S_X(f) \rightarrow \frac{2(E[\tau_a] + E[\tau_b])^{-1}}{(2\pi f)^2} \quad \text{as } f \rightarrow \infty. \quad (8.7)$$

In the special case when $p_{\tau_a}(t) = p_{\tau_b}(t) \equiv p_\tau(t)$, with $p_\tau(t)$ arbitrary, the spectrum set forth in Eq. (8.5) simplifies to (Aizawa, 1984)

$$S_X(f) = \frac{\delta(f)}{2} + \frac{(2\pi f)^{-2}}{E[\tau]} \operatorname{Re} \left\{ \frac{1 - \phi_\tau(2\pi f)}{1 + \phi_\tau(2\pi f)} \right\}. \quad (8.8)$$

We turn now to a special case in which there is extreme asymmetry in the dwell times, such that the times τ_b spent in state $X(t) = b$ greatly exceed the times τ_a spent in state $X(t) = a$. As shown in Sec. A.5.3, the spectrum then simplifies to a form closely related to that provided in Eq. (4.16) for the renewal point process (Lowen, 1992). More formally, given a randomly selected pair of dwell times τ_a and τ_b , and for frequencies $f \ll 1/E[\tau_a]$, the relation $\Pr\{\tau_a \ll \tau_b\} \approx 1$ implies that

$$S_X(f) \approx \frac{E[\tau_a]}{E[\tau_b]} \delta(f) + \frac{E[\tau_a^2]}{E[\tau_b]} \operatorname{Re} \left\{ \frac{1 + \phi_{\tau_b}(2\pi f)}{1 - \phi_{\tau_b}(2\pi f)} \right\}. \quad (8.9)$$

From a geometrical perspective, if $X(t) = a = 1$ only infrequently, then $X(t)$ remains in state $b = 0$ except for relatively brief visits to state $a = 1$. As examined in more detail in Sec. A.5.3, such an alternating renewal process $X(t)$ then resembles a renewal point process, albeit with thin, unit-height rectangles in place of the infinite-height delta functions that comprise the renewal point process. Under these circumstances, a marked version of the renewal point process provides a useful approximation to the alternating renewal process. The marks take values given by the times τ_a spent in the state $X(t) = a$, and are therefore independent and identically distributed in accordance with $p_{\tau_a}(t)$.

In fact, the alternating renewal process resembles the renewal point process by construction, so it is not surprising that they share many characteristics in common. The symmetric alternating renewal process and the renewal point process, by definition, have identical transition number statistics. They differ only in the type of transition; the former has two types that alternate. For the extreme asymmetric alternating renewal process, where τ_a almost always lies well below τ_b , as discussed above, a different correspondence exists. Here the state transitions occur in closely spaced pairs separated by negligible widths, so that the alternating renewal process transition-*pair* number statistics closely follow the renewal point-process *single*-transition number statistics. Moreover, the spectrum (and thus the coincidence rate) for the extreme asymmetric alternating renewal process is proportional to that for the renewal point process, except in the high-frequency (short-time) limit where the alternating-renewal-process spectrum varies as $S_X(f) \approx 2(2\pi f)^{-2}/E[\tau_b]$.

Finally, for a Markovian system where both dwell times follow exponential distributions, Rice (1944, 1945) demonstrated that the general result for the spectrum

provided in Eq. (8.5) assumes the familiar Lorentzian form,

$$S_X(f) = E[X] \delta(f) + \frac{2(E[\tau_a] + E[\tau_b])^{-1}}{(2\pi f)^2 + (2\pi f_S)^2}, \tag{8.10}$$

where $2\pi f_S \equiv 1/E[\tau_a] + 1/E[\tau_b]$.

8.2 ALTERNATING FRACTAL RENEWAL PROCESS

We turn now to the properties of the alternating fractal renewal process, a special alternating renewal processes in which one or both of the dwell-time distributions follows a fractal (power-law) form (see Fig. 8.1).

8.2.1 Spectrum

Consider the case when the dwell times in both states, τ_a and τ_b , have identical, abrupt-cutoff power-law distributions of the form displayed in Eq. (7.1). As shown in Sec. A.4.1, in the mid-frequency range ($A^{-1} \ll f \ll B^{-1}$) the alternating-fractal-renewal-process spectrum becomes (Lowen, 1992)

$$S_X(f) \rightarrow \frac{E[\mu]}{4} \times \begin{cases} 2\Gamma(1-\gamma) \cos(\pi\gamma/2) A^\gamma (2\pi f)^{\gamma-2} & 0 < \gamma < 1 \\ \frac{1}{2} A f^{-1} & \gamma = 1 \\ 2(\gamma-1)^{-1} \Gamma(2-\gamma) [-\cos(\pi\gamma/2)] A^\gamma (2\pi f)^{\gamma-2} & 1 < \gamma < 2 \\ 2A^2 [-\ln(2\pi f A)] & \gamma = 2 \\ \gamma(\gamma-2)^{-1} A^2 & \gamma > 2. \end{cases} \tag{8.11}$$

The spectrum $S_X(f)$ follows a power-law form, but the exponent in the range $0 < \gamma < 1$ in Eq. (8.11) differs from that in Eq. (7.8) for the renewal-point-process spectrum $S_N(f)$.

Equation (8.11) establishes that the fractal exponent α depends on γ in accordance with

$$\alpha = \begin{cases} 2-\gamma & 0 < \gamma < 2 \\ 0 & \gamma > 2. \end{cases} \tag{8.12}$$

This relationship differs from that provided in Eq. (7.9) for the fractal renewal point process; the fractal exponent α now extends over a larger range $0 < \alpha < 2$. The alternating fractal renewal process thus can serve as a source of $1/f^\alpha$ noise for $0 < \alpha < 2$, over the frequency range $B^{-1} \ll f \ll A^{-1}$ (see Prob. 8.6).

We present the normalized spectrum, $S_X(f)/S_0$, in Fig. 8.2. To facilitate comparisons among the plots, we divide by $S_0 \equiv \lim_{f \rightarrow 0} S_X(f)$ so that the normalized spectrum becomes unity at the low-frequency limit. The asymptotes are specified by Eqs. (8.6), (8.7), and (8.11), and the spectrum follows a $1/f^\alpha$ form over a range of frequencies. Taquq & Levy (1986) set forth a closely related process that has no upper cutoffs.

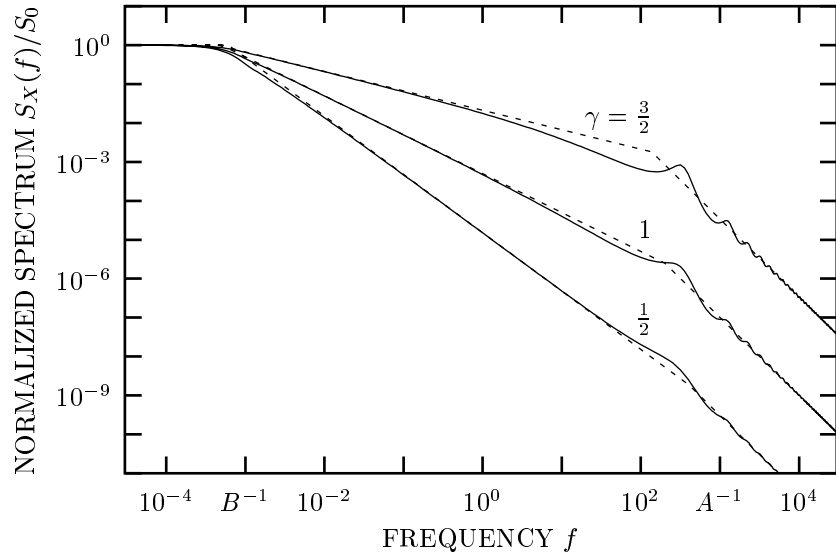


Fig. 8.2 Normalized spectrum $S_X(f)/S_0$ for the alternating fractal renewal process (solid curves). Both dwell times, τ_a and τ_b , derive from identical, abrupt-cutoff power-law distributions that take the form of Eq. (7.1), with $A = 10^{-3}$ and $B = 10^3$. The remaining parameter, the power-law exponent γ , takes the values $\frac{3}{2}$ (top), 1 (middle), and $\frac{1}{2}$ (bottom). Asymptotic forms from Eqs. (8.6), (8.11), and (8.7) highlight the low-, mid-, and high-frequency limits, respectively (dashed lines). All spectra decrease with frequency as $f^{-\alpha} = f^{\gamma-2}$ in the region $B^{-1} \ll f \ll A^{-1}$, in accordance with Eq. (8.11), and exhibit oscillations associated with the abrupt cutoffs in the probability densities.

8.2.2 Autocovariance

The autocovariance (defined as the autocorrelation minus the square of the mean), like the coincidence rate, can display power-law behavior as a result of the power-law-varying spectrum and the properties of the Fourier transform. For $1 < \gamma < 2$ in the region $A \ll |t| \ll B$, Eq. (8.11) leads to (Lowen & Teich, 1993d)

$$R_X(t) - E^2[X] \approx (4\gamma)^{-1} A^{\gamma-1} |t|^{1-\gamma}. \tag{8.13}$$

For $0 < \gamma \leq 1$ in the region $A \ll |t| \ll B$, on the other hand, the autocovariance does not exhibit power-law behavior.

8.2.3 1/f-type noise from Markov processes

A chain of Markov alternating renewal processes, with rates that scale geometrically, provides another method for generating a process with a 1/f-type spectrum (Lowen, Liebovitch & White, 1999). A variant of this construction imparts fractal behavior by embedding on-off processes endowed with different time scales, one inside another

(Krishnam, Venkatachalam & Capone, 2000). This latter construction begins with a largest-time-scale Markov alternating renewal process, with its “on” state corresponding to another Markov alternating renewal process with a shorter time scale, and so on, for a number of stages. In both constructions, the overall fractal behavior emerges from the combined scaling set of Markov processes.

8.3 BINOMIAL NOISE

We turn now to an analysis of the sum of M statistically identical and independent copies of the alternating renewal process. We begin by examining the amplitude distribution of the sum, which we denote X_Σ :

$$X_\Sigma(t) \equiv \sum_{n=1}^M X_n(t). \tag{8.14}$$

Figure 8.3 displays an example of the sum for $M = 4$ such processes.

Since the marginal distribution of each of the component processes follows a Bernoulli form with parameter

$$r \equiv \Pr\{X = 1\} = E[X] = \frac{E[\tau_a]}{E[\tau_a] + E[\tau_b]}, \tag{8.15}$$

as provided in Eq. (8.1), their sum comprises a binomial random variable with parameters r and M (Feller, 1968). The resulting process, $X_\Sigma(t)$, thus has a marginal amplitude that follows the **binomial distribution**:

$$\Pr\{X_\Sigma = n\} = \frac{M!}{n!(M-n)!} r^n (1-r)^{M-n}, \tag{8.16}$$

with moments and related quantities given by

mean	$E[X_\Sigma]$	=	Mr	
variance	$\text{Var}[X_\Sigma]$	=	$Mr(1-r)$	
skewness	$\frac{E[(X_\Sigma - E[X_\Sigma])^3]}{\text{Var}^{3/2}[X_\Sigma]}$	=	$\frac{1-2r}{\sqrt{Mr(1-r)}}$	(8.17)
kurtosis	$\frac{E[(X_\Sigma - E[X_\Sigma])^4]}{\text{Var}^2[X_\Sigma]} - 3$	=	$\frac{1}{M} \left[\frac{1}{r(1-r)} - 6 \right]$.	

We now examine the second-order properties of X_Σ , beginning with the autocorrelation. Using the independence of the component alternating renewal processes $X_n(t)$ that comprise $X_\Sigma(t)$, we obtain

$$R_{X_\Sigma}(t) \equiv E \left[\sum_{n=1}^M X_n(s) \sum_{m=1}^M X_m(s+t) \right]$$

$$\begin{aligned}
 &= \sum_{n=1}^M \sum_{m=1}^M E[X_n(s) X_m(s+t)] \\
 &= \sum_{n=1}^M \left\{ E[X_n(s) X_n(s+t)] + \sum_{m \neq n} E[X_n(s) X_m(s+t)] \right\} \\
 &= \sum_{n=1}^M \left\{ R_X(t) + \sum_{m \neq n} E[X] E[X] \right\} \\
 &= MR_X(t) + M(M-1) E^2[X], \tag{8.18}
 \end{aligned}$$

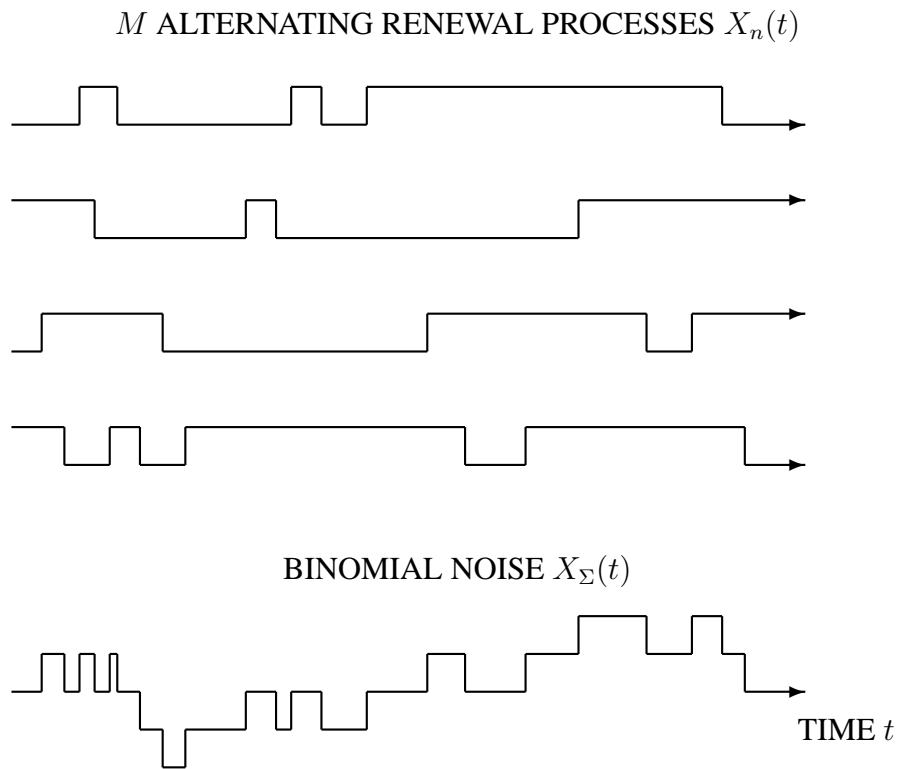


Fig. 8.3 The addition of M statistically independent and identical alternating renewal processes $X_n(t)$ leads to a process $X_\Sigma(t)$ with similar second-order characteristics but with a binomial amplitude distribution: *binomial noise*. We illustrate sample functions for four alternating renewal processes, along with their arithmetic sum, which is binomial noise. Since the binomial-noise process comprises all transitions that occur within each of the individual alternating renewal processes, it exhibits greater fluctuations than the constituent processes. The sum of alternating fractal renewal processes results in *fractal binomial noise*.

so that

$$R_{X_\Sigma}(t) - \mathbb{E}^2[X_\Sigma] = M\{R_X(t) - \mathbb{E}^2[X]\}. \quad (8.19)$$

Thus, the statistics of the binomial noise process $X_\Sigma(t)$ closely resemble those of the component alternating renewal processes $X_n(t)$. Because binomial noise exhibits all of the transitions occurring within each of the individual alternating renewal processes, as mentioned above, $X_\Sigma(t)$ exhibits M times more fluctuations per unit time than each of the constituent processes $X_n(t)$.

8.3.1 Fractal binomial noise

If the $X_n(t)$ exhibit fractal characteristics, and thus belong to the class of alternating fractal renewal processes, then $X_\Sigma(t)$ becomes fractal binomial noise. This process therefore has the same second-order statistics as its component alternating fractal renewal processes, and thus belongs to the fractal class of processes, as well as having a binomial amplitude distribution.

Finally, we mention an extension to the alternating fractal renewal process, in which $X(t)$ assumes random amplitudes during the on state, all independent and drawn from the same distribution (Yang & Petropulu, 2001) (see Prob. 8.5). The aggregation of a number of such processes provides a generalized form of fractal binomial noise, with similar temporal characteristics but arbitrary amplitude statistics.

8.3.2 Convergence to a Gaussian process

Consider again a sum of alternating renewal processes that gives rise to binomial noise, as set forth at the beginning of Sec. 8.3. As the number of constituent processes that comprise the sum increases, the skewness and kurtosis diminish, as shown in Eq. (8.17). Since the second moment of $X_n(t)$ exists, the central limit theorem applies and $X_\Sigma(t)$ approaches a Gaussian process as M increases. However, since the mean and variance of $X_\Sigma(t)$ increase without limit as M increases, we construct a process with bounded statistics for arbitrary values of M by using the usual linear conversion

$$X_M^*(t) \equiv [Mr(1-r)]^{-1/2} \sum_{n=1}^M [X_n(t) - Mr], \quad (8.20)$$

where r is the Bernoulli parameter set forth in Eq. (8.15). The quantity $X_M^*(t)$, as defined in Eq. (8.20), has zero mean by construction, and unit variance for all values of M , and thus remains well defined as $M \rightarrow \infty$. For delay times t in the range $A \ll t \ll B$, $X_M^*(t)$ has a covariance $R_{X_M^*}(t)$ proportional to that of the fractal Gaussian process discussed in Sec. 6.3.3. Since the mean and covariance define a Gaussian process, $X_M^*(t)$ indeed converges to a fractal Gaussian process.

We illustrate the convergence graphically in Fig. 8.4 for three values of the number of constituent alternating renewal processes M : 10, 100, and 1 000. The small number of processes results in a blocky appearance for the $M = 10$ curve; larger values of M lead to smoother plots. For simplicity of presentation, we employed Markov

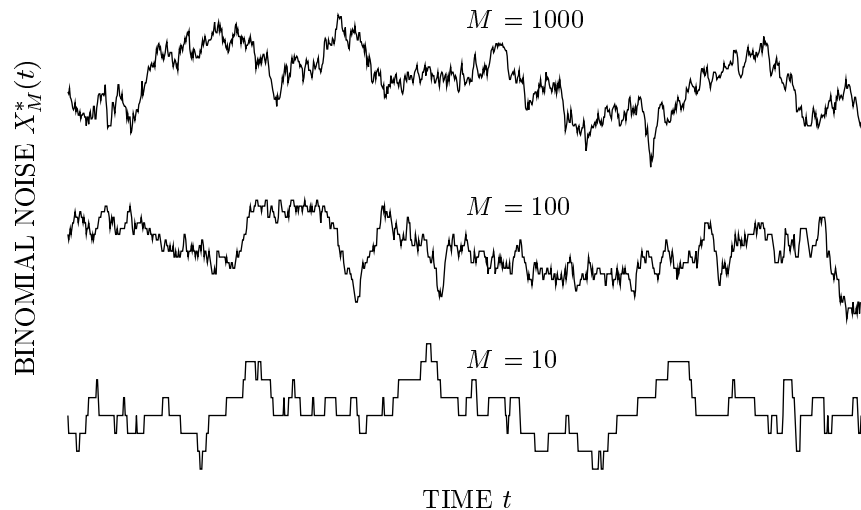


Fig. 8.4 Zero-mean unit-variance versions of binomial noise $X_M^*(t)$ constructed from sums of different numbers of constituent alternating renewal processes: $M = 10, 100,$ and $1\,000$. The plots are displaced by different amounts. For simplicity, the alternating renewal processes $X_n(t)$ all belong to the Markov family; each has a symmetric, exponentially decaying dwell-time density with a mean equal to $\frac{1}{10}$ of the horizontal extent of the graph. The discrete nature of the process is readily apparent for small numbers of constituent processes (bottom plot); it diminishes as the number increases (middle), and becomes undetectable at the resolution used for still larger numbers (top).

alternating renewal processes for this figure. Similar results obtain for fractal binomial noise over times longer than the upper cutoff B of the constituent alternating fractal renewal processes. Over significantly shorter times, however, most of the constituent processes do not change state, which leads to a proportional decrease in the apparent number of processes, independent of M . Within the scaling range $A \ll t \ll B$, this apparent number increases with the time t in a power-law fashion.

8.4 POINT PROCESSES FROM FRACTAL BINOMIAL NOISE

Since binomial noise (and, in particular, fractal binomial noise) has a well-defined, finite integral and never assumes negative values, it can directly serve as the rate for a point process. We briefly consider several point processes constructed from fractal binomial noise.

If the binomial noise modulates a Poisson process, we refer to the resulting doubly stochastic Poisson point process as a **fractal-binomial-noise-driven Poisson process**. The relations provided in Sec. 4.3, together with those set forth earlier in this chapter, yield the statistical properties of this point process. In particular, the spec-

trum differs from that of fractal binomial noise only by an additive constant so that the fractal features of the underlying alternating fractal renewal processes are preserved, at least up to the frequency at which the constant becomes important. In the limit where the binomial noise converges to a Gaussian process, as described in Sec. 8.3.2, the process converges to the **fractal-Gaussian-process-driven Poisson process**³ (see Secs. 6.3.3 and 10.6.1, as well as Fig. 5.5 and Chapter 12). The linearity of the Poisson transform yields relatively tractable results; however, it does introduce additional variability into the point process.

Transforms other than the Poisson can, of course, be implemented. If fractal binomial noise serves as the input to an integrate-and-reset process, instead of to a Poisson process, the **fractal-binomial-noise-driven integrate-and-reset process** results (see Sec. 4.4). The integrate-and-reset operation does not introduce additional variability so that the spectrum of the point process closely follows that of the fractal binomial noise, at least for sufficiently slow rate fluctuations. However, the highly nonlinear integrate-and-reset construct makes it difficult to obtain useful approximations in many regimes. As a final example, we mention the **fractal-binomial-noise-driven gamma process**, which has been considered as a model for characterizing sequences of action potentials in the mammalian visual system (Teich et al., 1997).

Problems

8.1 *Spectrum for symmetric alternating renewal process* Starting with Eq. (8.5), show that if τ_a and τ_b have identical density functions, the corresponding alternating renewal process has a spectrum given by Eq. (8.8).

8.2 *Asymptotic limits of Lorentzian spectrum* Show that the Lorentzian spectrum of Eq. (8.10) approaches the limits shown in Eqs. (8.6) and (8.7) for low and high frequencies, respectively.

8.3 *Spectrum from dwell-time characteristic functions* Demonstrate that the substitution of characteristic functions corresponding to exponentially distributed dwell times in Eq. (8.5) yields a spectrum that accords with Eq. (8.10).

8.4 *Power-law autocovariance from power-law spectrum* Show that Eq. (8.11) leads to Eq. (8.13).

8.5 *Alternation between arbitrary values* We specified for simplicity that the alternating fractal renewal process $X_1(t)$ switches between two particular values, zero and unity; however, other values can be used as well. Suppose we define a new process $X_2(t)$ that has the same transition times as $X_1(t)$, but assumes values of b and a instead of zero and unity, respectively.

8.5.1. Express $E[X_2^m]$ in terms of $E[X_1^m]$, for $0 < m \leq n$.

8.5.2. Express the autocorrelation $R_{X_2}(t)$ in terms of $R_{X_1}(t)$ and $E[X]$.

³ Although the fractal-binomial-noise-driven Poisson process is highly effective as a model in many applications, it is often replaced by this limiting form.

8.5.3. Now consider an alternating fractal renewal process $X_2(t)$ where a takes on random values, remaining the same during each “on” period, but independent of the values taken during other “on” periods (Yang & Petropulu, 2001). Set $b = 0$ for simplicity. What condition must be imposed on a in order that the autocorrelation $R_{X_2}(t)$ have a meaningful value for all times t ?

8.5.4. Assuming that this condition is satisfied, find an expression for $R_{X_2}(t)$.

8.6 *Digital generation of $1/f^\alpha$ noise* Simulation of the alternating fractal renewal process generates a synthetic signal with a spectrum that varies as $1/f^\alpha$. Given a random number X that is uniformly distributed in the interval $0 < X < 1$, the transformed random variable

$$Y = P_Y^{-1}(X) \quad (8.21)$$

has the probability distribution $P_Y(y)$, where P_Y^{-1} denotes the inverse of the function $P_Y(y)$.

8.6.1. Given a source of computer-generated random numbers $\{X_U\}$ uniformly distributed in the unit interval ($0 < X_U < 1$), show how to generate a test signal with a $1/f^\alpha$ spectrum over the range $f_L \ll f \ll f_H$.

8.6.2. A Gaussian-distributed test signal generally proves more useful than one with only two states, such as that considered above. Suppose now that there is a further constraint requiring that both the skewness and kurtosis assume values within ϵ of those for a Gaussian random variable. To this end, we construct fractal binomial noise by summing the values of M of these processes. We may also employ different values of A or B for the dwell times in the two states to generate an asymmetric process. What values for the asymmetry and for M satisfy these constraints with the smallest number of processes M ?